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OBSERVATIONS WEIGHTED ACCORDING TO ORDER.

BY P. J. DANIELL.

1. *Introduction.*—When a series of measurements of some quantity are made, two particular quantities require to be calculated expressing respectively the norm and the deviation. For the norm the mean or the median is used while there are three measures of dispersion, the standard or root-mean-square deviation, the mean numerical deviation and the quartile deviation. The question is as to which of these are the more accurate under a general law. Moreover if we choose for our norm the mean or average it appears occasionally profitable to discard one or several extreme measures. Whether, or in what cases, this is legitimate is discussed by Poincaré* but no general conclusions are obtained.

Besides such a discard-average we might invent others in which weights might be assigned to the measures according to their order. In fact the ordinary average or mean, the median, the discard-average, the numerical deviation (from the median, which makes it minimum), and the quartile deviation can all be regarded as calculated by a process in which the measures are multiplied by factors which are functions of order. It is the general purpose of this paper to obtain a formula for the mean square deviation of any such expression. This formula may then be used to measure the relative accuracies of all such expressions.

Certain particular types are discussed and their accuracies calculated in percentages.

Unfortunately the standard deviation is not of the same general type and therefore we add a note on its accuracy. The assumptions made are fairly general. On the one hand the number of observations, n, is supposed large and terms of order higher than $1/n$ are discarded; on the other the probability law assumed is regular and indefinitely differentiable. In our applications to special types, however, we shall only consider cases in which the theoretical distribution is symmetrical, and this for logical reasons. It is useless to compare the relative merits of the various kinds of average, for example, the mean and the median, unless they all tend to coincide when n increases indefinitely. If there is a lack of symmetry both the mean and the median are necessary, or at least valuable, indications of the nature of the distribution. Indeed, in practise, their difference is sometimes regarded as a measure of lack of symmetry.

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* Poincaré, "Calcul des Probabilités" (1912), p. 211.

2. *Mathematical Analysis.*—Assume that n measurements $t_1, t_2, \cdots, t_n$ are made and that their magnitudes are in the order of their suffixes, so that

$$t_1 \leqq t_2, \text{ and so on.}$$

Multiply by the factors $f_1, f_2, \cdots, f_n$, so that

$$\bar{t} = \sum_{r=1}^{n} f_r t_r.$$

We desire to find a formula for the mean square deviation of $\bar{t}$ when the measurements, t_r, are subject to some law of probability $p(t)$.

If $\varphi(t_1, \cdots, t_n)$ is some function of the measures considered in their proper order, the average value of φ when $t_1, \cdots, t_n$ vary according to the law of probability will be denoted by Av (φ) to distinguish this from the weighted average, $\bar{t}$, which we obtain for a particular fixed set of values $t_1, t_2, \cdots, t_n$.

Allowing for the possible permutations of the suffixes,

$$\text{Av}(\varphi) = n! \int_{-\infty}^{+\infty} p(t_n) dt_n \int_{-\infty}^{t_n} p(t_{n-1}) dt_{n-1} \cdots \int_{-\infty}^{t_2} p(t_1)\varphi(t_1, \cdots, t_n) dt_1.$$

If

$$\int_{-\infty}^{t} p(t)dt = x,$$

let $t = t(x)$; then x varies from 0 to 1, and

$$\text{Av}(\varphi) = n! \int_0^1 dx_n \int_0^{x_n} dx_{n-1} \cdots \int_0^{x_2} \psi(x_1, \cdots, x_n) dx_1, \tag{1}$$

where

$$\psi(x_1, \cdots, x_n) = \varphi[t(x_1), \cdots, t(x_n)].$$

We shall make frequent use of the formula

$$\int_0^a dx_p \int_0^{x_p} dx_{p-1} \cdots \int_0^{x_1} f(x)dx = \frac{1}{p!}\int_0^a f(x)(a-x)^p dx. \tag{2}$$

This formula can readily be verified by differentiating with respect to a. A particular case is that in which $f(x) = 1$,

$$\int_0^a dx_p \int_0^{x_p} dx_{p-1} \cdots \int_0^{x_1} dx = \frac{1}{p!}\int_0^a (a-x)^p dx = \frac{1}{(p+1)!}a^{p+1}. \tag{2a}$$

Substituting from (2*a*) in (1)

$$\text{Av}(1) = n! \int_0^1 dx_n \int_0^{x_n} dx_{n-1} \cdots \int_0^{x_2} dx_1 = n! \cdot \frac{1}{n!} \cdot 1^n = 1.$$

This confirms the coefficient $n!$ in the formula (1).

$$\begin{aligned}
\mathrm{Av}\,(t_r) &= n!\int_0^1 dx_n\int_0^{x_n} dx_{n-1}\cdots\int_0^{x_2} t(x_r)dx_1\\
&= n!\int_0^1 dx_n\int_0^{x_n} dx_{n-1}\cdots\int_0^{x_{r+1}} t(x_r)\frac{x_r^{r-1}}{(r-1)!}dx_r \quad [\text{by } 2a]\\
&= \frac{n!}{(n-r)!\,(r-1)!}\int_0^1 t(x)(1-x)^{n-r}x^{r-1}dx. \qquad (3)
\end{aligned}$$

When r, n are large the integrand will have a steep maximum near $x = r/n$. Also

$$\frac{n!}{(n-r)!\,(r-1)!}\int_0^1 x^p(1-x)^{n-r}x^{r-1}dx = \frac{r}{n+1}\cdot\frac{r+1}{n+2}\cdots\frac{r+p-1}{n+p}.$$

Denote $r/(n+1)$ by x_r and neglect terms of order higher than $1/n$.

$$\frac{r+1}{n+2} = x_r + \frac{1}{n}(1-x_r), \qquad \frac{r+2}{n+3} = x_r + \frac{2}{n}(1-x_r), \qquad \text{etc.}$$

$$\begin{aligned}
&\frac{n!}{(n-r)!\,(r-1)!}\int_0^1 x^p(1-x)^{n-r}x^{r-1}dx\\
&\qquad = x_r\left[x_r+\frac{1}{n}(1-x_r)\right]\cdots\left[x_r+\frac{p-1}{n}(1-x_r)\right]\\
&\qquad = x_r{}^p + \frac{p(p-1)}{2n}x_r{}^{p-1}(1-x_r).
\end{aligned}$$

$$\begin{aligned}
&\frac{n!}{(n-r)!\,(r-1)!}\int_0^1 (x-x_r)^p(1-x)^{n-r}x^{r-1}dx\\
&\qquad = \sum_{q=0}^{p}\frac{p!}{q!(p-q)!}(-1)^q x_r^p\\
&\qquad\quad + \sum_{q=0}^{p}\frac{p!}{q!(p-q)!}(-1)^q\frac{(p-q)(p-q-1)}{2n}x_r^{p-1}(1-x_r).
\end{aligned}$$

Of these two sums the former is 0 unless $p = 0$ and the latter is 0 unless $p = 2$.

$$\begin{aligned}
\frac{n!}{(n-r)!\,(r-1)!}\int_0^1 (x-x_r)^p(1-x)^{n-r}x^{r-1}dx &= 0 \qquad (p \neq 0, 2)\\
&= 1 \qquad (p = 0)\\
&= \frac{1}{n}x_r(1-x_r) \qquad (p = 2).
\end{aligned}$$

[The reader is reminded that these equations are satisfied only as far as terms of order $1/n$.]

Expand $t(x)$ by a Taylor development near $x = x_r$,

$$t(x) = t(x_r) + (x - x_r)t'(x_r) + \frac{(x - x_r)^2}{2!} t''(x_r) + \cdots.$$

Substitute into (3) and use the formula just obtained, then

$$\text{Av}\ (t_r) = t(x_r) + \frac{1}{2n} x_r(1 - x_r)t''(x_r). \tag{4}$$

By the same reasoning,

$$\begin{aligned} \text{Av}\ (t_r^2) &= t^2(x_r) + \frac{1}{2n} x_r(1 - x_r)[2t(x_r)t''(x_r) + 2\{t'(x_r)\}^2] \\ &= [\text{Av}\ (t_r)]^2 + \frac{1}{n} x_r(1 - x_r)[t'(x_r)]^2. \end{aligned} \tag{5}$$

We next require to calculate Av $(t_r t_s)$ and must agree on order. Suppose $s > r$, then

$$\begin{aligned} \text{Av}\ (t_r t_s) &= n! \int_0^1 dx_n \int_0^{x_n} dx_{n-1} \cdots \int_0^{x_2} t(x_r)t(x_s)dx_1 \\ &= \frac{n!}{(s - r - 1)!\,(r - 1)!} \int_0^1 dx_n \int_0^{x_n} dx_{n-1} \\ &\qquad \cdots \int_0^{x_{s+1}} t(x)dx \int_0^x t(y)(x - y)^{s-r-1}y^{r-1}dy \qquad (6) \\ &= \frac{n!}{(n - s)!\,(s - r - 1)!\,(r - 1)!} \\ &\qquad \times \int_0^1 (1 - x)^{n-s}t(x)dx \cdot \int_0^x (x - y)^{s-r-1}y^{r-1}t(y)dy. \end{aligned}$$

In this double integral the integrand has a steep maximum near

$$x = s/n, \qquad y = r/n.$$

$$\begin{aligned} &\frac{n!}{(n - s)!\,(s - r - 1)!\,(r - 1)!} \int_0^1 x^p(1 - x)^{n-s}dx \int_0^x y^q(x - y)^{s-r-1}y^{r-1}dy \\ &= \frac{r}{n + 1} \cdot \frac{r + 1}{n + 2} \cdots \frac{r + q - 1}{n + q} \cdot \frac{s + q}{n + q + 1} \cdots \frac{s + q + p - 1}{n + q + p}. \end{aligned}$$

Denote $r/(n + 1)$ by x_r, $s/(n + 1)$ by x_s and expand as far as $1/n$,

$$\frac{r + 1}{n + 2} = x_r + \frac{1}{n}(1 - x_r), \qquad \text{etc.,}$$

$$\frac{s + q}{n + q + 1} = x_s + \frac{q}{n}(1 - x_r), \qquad \text{etc.}$$

$$\frac{n!}{(n-s)!\,(s-r-1)!\,(r-1)!}\int_0^1 x^p(1-x)^{n-s}dx\int_0^x y^q(x-y)^{s-r-1}y^{r-1}dy$$

$$= x_r^q x_s^p + \frac{q(q-1)}{2n}\,x_r^{q-1}(1-x_r)x_s^p + \frac{p(2q+p-1)}{2n}\,x_r^q x_s^{p-1}(1-x_s)$$

$$= x_r^q x_s^p + \frac{q(q-1)}{2n}\,x_r^{q-1}(1-x_r)x_s^p + \frac{p(p-1)}{2n}\,x_s^{p-1}(1-x_s)x_r^q$$

$$+ \frac{pq}{n}\,x_r^{q-1}x_s^{p-1}x_r(1-x_s).$$

Using a method similar to that given above,

$$\frac{n!}{(n-s)!\,(s-r-1)!\,(r-1)!}$$

$$\times\int_0^1 (x-x_s)^p(1-x)^{n-s}dx\int_0^x (y-x_r)^q(x-y)^{s-r-1}y^{r-1}dy = 0,$$

except for $p = 0,\ q = 0;\ p = 0,\ q = 2;\ p = 2;\ q = 0;\ p = 1,\ q = 1.$

In formula (6) expand

$$t(x) = t(x_s) + (x-x_s)t'(x_s) + \cdots, \qquad t(y) = t(x_r) + (y-x_r)t'(x_r) + \cdots,$$

then by similar reasoning as before

$$\text{Av }(t_r t_s),\ (s > r) = t(x_r)t(x_s) + \frac{1}{2n}x_r(1-x_r)t''(x_r)t(x_s)$$

$$+ \frac{1}{2n}x_s(1-x_s)t(x_r)t''(x_s) + \frac{1}{n}x_r(1-x_s)t'(x_r)t'(x_s) \tag{7}$$

$$= \text{Av }(t_r)\cdot\text{Av }(t_s) + \frac{1}{n}x_r(1-x_s)t'(x_r)t'(x_s).$$

Now

$$\bar{t} = \sum_{r=1}^{n} f_r t_r$$

$$\text{Av }(\bar{t}) = \sum_{r=1}^{n} f_r\,\text{Av }(t_r),$$

$$\text{Av }(\bar{t}^2) = \sum_{r=1}^{n} f_r^2\,\text{Av }(t_r^2) + 2\sum_{r=1}^{n}\ \sum_{s=r+1}^{n} f_r f_s\,\text{Av }(t_r t_s)$$

[By 5, 7].

$$= \sum_{r=1}^{n} f_r^2[\text{Av }(t_r)]^2 + 2\sum_{r=1}^{n}\ \sum_{s=r+1}^{n} f_r f_r\,\text{Av }(t_r)\cdot\text{Av }(t_s)$$

$$+ \sum_{r=1}^{n} f_r^2\,\frac{x_r(1-x_r)}{n}[t'(x_r)]^2$$

$$+ 2\sum_{r=1}^{n}\ \sum_{s=r+1}^{n} f_r f_r\,\frac{x_r(1-x_s)}{n}\,t'(x_r)t'(x_s)$$

Let $S^2 = n \times x$ mean square deviation of $(\bar{t}) = n[\mathrm{Av}\,(\bar{t}^2) - \mathrm{Av}^2\,(\bar{t})]$. Then let

$$f(x_r) = nf_r, \qquad f(x_r)\Delta x = f(x_r)\cdot\frac{1}{n} = f_r,$$

and replace the double sum by an integral.

$$S^2 = 2\int_0^1 f(x)(1-x)t'(x)dx\int_0^x f(y)yt'(y)dy.$$

Let

$$\varphi(t) = \int_c^t f[x(t)]dt,$$

where c is so chosen that

$$\int_{-\infty}^{+\infty}\varphi(t)p(t)dt = 0. \tag{8}$$

Let

$$\psi(x) = \varphi[t(x)], \qquad \psi'(x) = f(x)t'(x).$$

$$S^2 = 2\int_0^1 (1-x)\psi'(x)dx\int_0^x y\psi'(y)dy.$$

Consider the function

$$F(a, b) = 2\int_a^b (b-x)\psi'(x)dx\int_a^x (y-a)\psi'(y)dy,$$

$$\frac{\partial F}{\partial b} = 2\int_a^b \psi'(x)dx\int_a^x (y-a)\psi'(y)dy,$$

$$\frac{\partial^2 F}{\partial a\partial b} = -2\int_a^b \psi'(x)dx\int_a^x \psi'(y)dy = -[\psi(b) - \psi(a)]^2.$$

Integrating again and since $\partial F/\partial b = 0$, $F = 0$ when $a = b$,

$$\frac{\partial F}{\partial b} = \int_a^b [\psi(b) - \psi(y)]^2 dy, \qquad \frac{dF(0, b)}{db} = \int_0^b [\psi(b) - \psi(y)]^2 dy,$$

$$F(0, b) = \int_0^b dx\int_0^x dy[\psi(x) - \psi(y)]^2,$$

$$\begin{aligned} S^2 &= F(0, 1) \\ &= \int_0^1 dx\int_0^x dy[\psi(x) - \psi(y)]^2 \\ &= \int_{-\infty}^{+\infty} p(t)dt\int_{-\infty}^t p(u)du[\varphi(t) - \varphi(u)]^2. \end{aligned}$$

Interchange the order of integration and also the symbols t, u.

$$S^2 = \int_{-\infty}^{+\infty} p(t)dt\int_t^{\infty} p(u)du[\varphi(u) - \varphi(t)]^2.$$

Combining both forms,

$$S^2 = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}[\varphi(t) - \varphi(u)]^2 p(t)p(u)dtdu$$

$$= \int_{-\infty}^{+\infty}\varphi^2(t)p(t)dt - \left[\int_{-\infty}^{+\infty}\varphi(t)p(t)dt\right]^2.$$

But by (8) the last term is 0; then

$$S^2 = \int_{-\infty}^{+\infty}\varphi^2(t)p(t)dt. \tag{9}$$

This is the formula we set out to obtain.

3. *Norm and Deviation.*—For the norm or average $\bar{t} = \sum_r f_r t_r$, with the condition $\sum_{r=1}^{n} f_r = 1$.

Expressing this by the approximate integral and then integrating by parts,

$$\int_{-\infty}^{+\infty} f(t)p(t)dt = 1,$$

$$\int_{-\infty}^{+\infty}[-p'(t)]\varphi(t)dt = 1. \tag{10}$$

The mean is obtained by equal weighting, $f(t) = 1$, $\varphi(t) = t - t_0$, where t_0 is the theoretical average. Then

$$S^2 = \int_{-\infty}^{+\infty}(t - t_0)^2 p(t)dt = \sigma^2.$$

Then the mean square deviation of the mean of n measurements is

$$S^2/n = \sigma^2/n.$$

This particular result is well-known but it confirms our formula (9).

If several groups of measures are to be combined, the average from each group should be multiplied by a factor inversely as the square of the deviation in that group. If then we agree to take the accuracy of the mean as a standard, equal to 1 or 100 per cent., the accuracy of any norm will be measured by the ratio σ^2/S^2.

Definition.—The accuracy of a norm is defined to be σ^2/S^2, where σ^2 is the theoretical square deviation.

In the case of the measure of deviation condition (10) no longer applies but we must suppose the weights f_r chosen so that the average value of the deviation has a fixed value, D.

$$\text{Av}\,(\bar{t}) = \sum_{r=1}^{n} f_r\,\text{Av}\,(t_r) = D.$$

Expressing in integral form, and integrating by parts,

$$\begin{aligned} D &= \int_{-\infty}^{+\infty} tf(t)p(t)dt \\ &= \int_{-\infty}^{+\infty} \frac{d}{dt}(-tp)\varphi(t)dt \qquad \text{[By (4)]} \\ &= \int_{-\infty}^{+\infty} [-tp'(t)]\varphi(t)dt - \int_{-\infty}^{+\infty} p(t)\varphi(t)dt. \end{aligned}$$

Then, by (8),

$$\int_{-\infty}^{+\infty} [-tp'(t)]\varphi(t)dt = D. \tag{11}$$

For the measure of deviation condition (11) takes the place of (10).

Again if we double the value of D, by doubling f_r, we shall multiply S^2 by 4. A true measure of accuracy will be some multiple of D^2/S^2, and for reasons which appear later we make the

Definition.—The accuracy of a measure of deviation is defined to be $D^2/(2S^2)$, where D is the theoretical average deviation.

Standard Deviation.—The standard deviation may be defined as D where

$$D^2 = \frac{1}{n}\sum_{r=1}^{n} t_r^2 - \frac{1}{n^2}\left(\sum_{r=1}^{n} t_r\right)^2.$$

It is difficult if not impossible to obtain a formula, in the general case, for the average value of D; nevertheless, if the number n is large, the proportional error in D will be small of order $1/\sqrt{n}$. We have the right, therefore, to assume that the proportional error in D will be one half that in D^2. Then if

$$D' = D^2, \qquad S'^2 = n \times \text{mean square deviation of } D',$$

$$\frac{D^2}{2S^2} = \frac{2D'^2}{S'^2}.$$

Choose the origin for t so that

$$\text{Av}(t) = \int_{-\infty}^{+\infty} tp(t)dt = 0.$$

Let

$$\int_{-\infty}^{+\infty} t^2 p(t)dt = \sigma^2, \qquad \int_{-\infty}^{+\infty} t^4 p(t)dt = q^4.$$

Then

$$\begin{aligned} \text{Av}(t_r) &= \int t_r p(t_r)dt_r \int^{(n+1)} \cdots \int p(t_1)\cdots p(t_n)dt_1\cdots dt_n \\ &= \int tp(t)dt = 0. \end{aligned}$$

$$\text{Av}(t_r t_s) = \text{Av}(t_r t_s^3) = 0, \qquad \text{Av}(t_r^2) = \sigma^2,$$

$$\text{Av}(t_r^4) = q^4, \qquad \text{Av}(t_r^2 t_s^2) = \sigma^2 \cdot \sigma^2 = \sigma^4.$$

But

$$D' = \frac{1}{n}\sum_{r=1}^{n} t_r^2 - \frac{1}{n^2}\left(\sum_{r=1}^{n} t_r\right)^2 .$$

The only terms in D' and D'^2 which yield integrals different from 0 will be of the types t_r^2, $t_r^2 t_s^2$, t_r^4.

$$\text{Av } (D') = \frac{1}{n}(n\sigma^2) - \frac{1}{n^2}(n\sigma^2) = \frac{n-1}{n}\sigma^2.$$

$$\text{Av } (D'^2) = \frac{1}{n^2}\left[nq^4 + 2\frac{n(n-1)}{2}\sigma^4\right] - \frac{2}{n^3}\left[nq^4 + 2\frac{n(n-1)}{2}\sigma^4\right]$$
$$+ \frac{1}{n^4}\left[nq^4 + 6\frac{n(n-1)}{2}\sigma^4\right]$$
$$= \frac{(n-1)^2}{n^3}q^4 + \frac{n-1}{n^3}(n^2 - 2n + 3)\sigma^4.$$

$$S'^2 = n[\text{Av } (D'^2) - \{\text{Av } (D')\}^2] = \left(\frac{n-1}{n}\right)^2 q^4 - \frac{(n-1)(n-3)}{n^2}\sigma^4.$$

Omitting terms in $1/n$, $1/n^2$,

$$S^2 = \frac{1}{4}D^2 S'^2 \div D'^2 = \frac{D^2}{4}\frac{q^4 - \sigma^4}{\sigma^4}. \tag{12}$$

This formula gives the value of n times the mean square deviation of the standard deviation D.

When the theoretical distribution is normal or Gaussian,

$$q^4 = 3\sigma^4, \qquad S^2 = \frac{D^2}{4}\frac{3-1}{1} = \frac{D^2}{2}.$$

By the definition, the accuracy will be

$$\frac{D^2}{2S^2} = 1 = 100 \text{ per cent.}$$

This explains the factor 2 which is introduced to make the accuracy of the standard deviation 1 when the law is normal.

It is an interesting fact that the formula (12) proved for the standard deviation is the same as the corresponding value given by (9), when, instead of the mean-root-square deviation, we multiply every measurement by the theoretical value of t corresponding to its order in the series. For then

$$f(t) = \lambda t, \qquad \varphi(t) = \tfrac{1}{2}\lambda t^2 + \text{constant.}$$

This constant is chosen so that Av $(\varphi) = 0$, or

$$\varphi(t) = \tfrac{1}{2}\lambda(t^2 - \sigma^2).$$

Using the condition which led up to (11),

$$\lambda\sigma^2 = D.$$

From (9)

$$S^2 = \frac{\lambda^2}{4}\int_{-\infty}^{+\infty}(t^2 - \sigma^2)^2 p(t)dt$$

$$= \frac{\lambda^2}{4}(q^4 - 2\sigma^4 + \sigma^4) = \frac{D^2}{4}\,\frac{q^4 - \sigma^4}{\sigma^4}.$$

4. *Most Accurate Weighting.*—For the norm it is required to make S^2 given by formula (9) a minimum under condition (10). Then

$$\varphi_1(t) = \lambda_1\left[-\frac{p'(t)}{p(t)}\right], \qquad f_1(t) = \lambda_1\frac{d}{dt}\left[-\frac{p'(t)}{p(t)}\right],$$

where from (10)

$$\frac{1}{\lambda_1} = \int_{-\infty}^{+\infty}\frac{p'^2}{p}dt, \qquad S^2 = \lambda_1.$$

Accuracy

$$A_1 = \frac{\sigma^2}{S^2} = \sigma^2\int_{-\infty}^{+\infty}\frac{p'^2}{p}dt.$$

For the deviation, S^2 given by (9) is minimum under condition (11).

$$\varphi_2(t) = \lambda_2\left(-t\frac{p'}{p} - 1\right), \qquad f_2(t) = \lambda_2\frac{d}{dt}\left(-t\frac{p'}{p}\right),$$

where from (11)

$$\frac{D}{\lambda_2} = \int_{-\infty}^{+\infty}t^2\frac{p'^2}{p}dt - 1, \qquad S^2 = \lambda_2 D = D^2\frac{\lambda_2}{D}.$$

Accuracy

$$A_2 = \frac{D^2}{2S^2} = \frac{1}{2}\left[\int_{-\infty}^{+\infty}t^2\frac{p'^2}{p}dt - 1\right].$$

For the normal law,

$$-\frac{p'}{p} = \frac{t}{\sigma^2}, \qquad \frac{1}{\lambda_1} = \frac{1}{\sigma^2}, \qquad \frac{D}{\lambda_2} = 2.$$

$$f_1(t) = 1, \qquad f_2(t) = Dt/\sigma^2, \qquad A_1 = 1, \qquad A_2 = 1.$$

Thus for the normal law the most accurate norm is the equal-weighted average and the most accurate deviation that obtained by multiplying each measure by the algebraic theoretical deviation corresponding to its order. As we pointed out before the accuracy of the latter is the same as that of the standard deviation itself.

For the symmetric Pearson law,

$$-\frac{p'}{p} = \frac{2nt}{a^2 \pm t^2},$$

$$f_1(t) = 2\lambda_1 n\frac{a^2 \mp t^2}{(a^2 \pm t^2)^2}, \qquad f_2(t) = \frac{4\lambda_2 n a^2 t}{(a^2 \pm t^2)^2}.$$

If the distribution is supernormal, that is if the number of extreme cases is more than normal the weights for the norm and the weights ÷ t for the deviation should diminish outwards and for the norm should even become negative for large values of t.

On the other hand, if the distribution is subnormal the weights should increase and become infinite at the boundaries $t = \pm a$. In these cases the weighting to be applied is much too complicated to be of any practical value, aside from the impossibility of knowing beforehand the proper values of a and n. However the most important cases are supernormal rather than the reverse and instead of letting the weights diminish according to a complex law we may take equal weights, for the norm, up to a certain point and then discard all measures outside these limits. Such a norm we shall call a discard-average and in practise a certain outer fraction of the measures is discarded. For the deviation we may discard not the outer but the inner fraction. Our next paragraph deals with such special types.

5. *Special Types of Average.—Discard Average.* Let k be the central fraction of the group retained and let t_1 be the solution of

$$2\int_0^{t_1} p(t)dt = k. \tag{13}$$

Then

$$\begin{aligned} f(t) &= \frac{1}{k}, & -t_1 < t < +t_1 \\ &= 0, & \text{otherwise.} \\ \varphi(t) &= \frac{1}{k}t, & 0 < t < t_1 \\ &= \frac{1}{k}t_1, & t > t_1. \end{aligned}$$

By formula (9)

$$S^2 = \frac{1}{k^2}\left[2\int_0^{t_1} t^2p(t)dt + t_1^2 2\int_{t_1}^{\infty} p(t)dt\right].$$

Let α denote the ratio $2\int_0^{t_1} t^2p(t)dt : kt_1^2$,

$$2\int_0^{t_1} t^2p(t)dt = \alpha k t_1^2, \tag{14}$$

$$S^2 = \frac{t_1^2}{k^2}[1 - (1 - \alpha)k]. \tag{15}$$

In the case of the median k and t_1 approach 0 together, and

$$S^2 = \frac{1}{4p_0^2}. \tag{15a}$$

In the following discussion for purposes of comparison we shall use the normal Gaussian law and the two extreme Pearson symmetric forms,

$$\text{supernormal,} \qquad p(t) = \frac{2}{\pi}(1 + t^2)^{-2}, \tag{a}$$

$$\text{normal,} \qquad p(t) = \frac{1}{\sqrt{\pi}} e^{-t^2}, \tag{b}$$

$$\text{subnormal,} \qquad p(t) = \frac{3}{4}(1 - t^2), \qquad (t^2 \leqq 1). \tag{c}$$

The accuracy of the median will be, in percentages,

(*a*) 162, (*b*) 63.7, (*c*) 45.

For the quartile-discard average, $k = 1/2$ and the fraction α takes the values,

(*a*) .306, (*b*) .314, (*c*) .323.

Formula (15) becomes

$$S^2 = 2t_1^2(1 + \alpha).$$

The accuracy will be, in percentages,

(*a*) 200,* (*b*) 83.7, (*c*) 63.*

In a supernormal Pearson distribution,

$$p(t) = c(1 + t^2)^{-n}.$$

For the quartile-discard average, when n is large,

$$S^2 = \frac{1.195}{2n}\left[1 + \frac{.851}{n} + \frac{.700}{n^2}\right] = \frac{1.195}{2n - 1.70}.$$

Hence the accuracy of the quartile-discard average will be

$$A = 83.7\frac{2n - 1.70}{2n - 3.00}\text{ per cent.} = 83.7 + \frac{109}{2n - 3}\text{ per cent.}$$

The formula can hardly be used with accuracy when n is as small as 2, but even then it would give the value

$$A = 193.$$

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* These values are only rough approximations.

The quartile-discard average will be more accurate than the ordinary mean if

$$2n < 9.3.$$

If σ^2 is average t^2 and q^4 average t^4 then the above condition may be translated into

$$q^4 > 4{\cdot}4\sigma^4,$$

instead of $3\sigma^4$ for a normal law.

Median-Quartile Average.—This average is the mean of the median and the two quartiles,

$$\bar{t} = \frac{1}{3}(Q_1 + M + Q_3), \qquad S^2 = \frac{1}{36}\left(\frac{1}{p_0^2} + \frac{2}{p_0 p_1} + \frac{2}{p_1^2}\right),$$

where $p_0 = p(t_0)$, $p_1 = p(t_1)$ and t_1 is the theoretical quartile deviation.

For a normal law the accuracy is

$$A = 86.0 \text{ per cent.}$$

It appears to be a little more accurate than the quartile-discard average, but we have assumed that the number of observations is large. When the number is small it will be difficult to determine the quartiles exactly, so that, taking everything into consideration, we may say the median-quartile average and the quartile-discard average are about equally accurate.

The most serious objection to the use of any special type of average is that discontinuity is introduced; that is, if the measures are considered as sufficiently normal, none will be discarded; if not, some may be discarded and there will be a finite change in the average. To obviate this difficulty we might use a combination of quartile-discard and ordinary average.

Let p be the weight assigned to the ordinary average.

$q = 1 - p =$ weight assigned to the quartile-discard.

$\sigma^2 =$ mean square deviation.

$N =$ mean numerical deviation.

$P =$ quartile deviation, or probable error.

Then

$$S^2 = p^2\sigma^2 + 2pqP(2N - \tfrac{1}{6}D) + q^2\,\tfrac{8}{3}\,P^2, \tag{16}$$

approximately, assuming average t^2 from 0 to P is $\frac{1}{3}P^2$, average t from 0 to P is $\frac{1}{2}P$. Then we may choose p, q so as to make S^2 minimum.

6. *Special Types of Dispersion Measure. Numerical Deviation.*—In this

case

$$\begin{aligned} f(t) &= +1, \qquad t > 0, \\ &= -1, \qquad t < 0, \end{aligned}$$

$$\varphi(t) = |t| - N,$$

where N is the theoretical mean numerical deviation. Then, by (9),

$$S^2 = \sigma^2 - N^2.$$

By the definition, succeeding formula 11, the accuracy will be

$$\frac{N^2}{2S^2} = \frac{N^2}{2(\sigma^2 - N^2)} \cdot$$

In the case of our three Pearson types (*a*) supernormal, (*b*) normal, (*c*) subnormal, the accuracy, in percentages, will be

(*a*) 34, (*b*) 87.6, (*c*) 118.

Discard Deviation.—In this case we discard the inner portion and then use the mean numerical deviation of the remainder. Under a normal law, if the portion between the quartiles, that is the central half, is discarded the accuracy is 96.3 per cent. Hence this is practically as accurate as the standard deviation and may, in some cases, be more rapidly found as it is a numerical mean and the calculations are made for half the measures only.

Quartile Deviation, etc.—If t is the theoretical deviation the accuracy will be

$$8[tp(t)]^2.$$

For the quartile deviation this is 36.7 per cent.

It will be a maximum when $tp(t)$ is maximum and for a normal law this is given by $t = \sigma$. The values $t = \pm \sigma$ practically divide the whole range of measures into two thirds within and one third without.

We may call this the sextile deviation, remembering that it is the outermost sextiles only which are used. Then the accuracy becomes 46.8 per cent. Furthermore it is much easier to find the corresponding standard deviation σ in this case, for theoretically with a normal law

$$\sigma = \left(1 + \frac{1}{30}\right) \text{outer-sextile deviation.}$$

We might also call this the semi-probable error, that is, such that the chances of exceeding it are just one half the chances of not exceeding.

A table is added of the accuracies of the various types when the law is assumed to be normal.

Norms:

Median	63.7	per cent.
Quartile average $(Q_1 + M + Q_3)/3$	86.0	" "
Quartile discard average (outer quartiles discarded)	83.7	" "
Mean	100	" "

Dispersion:

Quartile deviation	36.7	" "
Outer sextile deviation	46.8	" "
Numerical deviation	87.6	" "
Discard deviation (inner quartiles discarded)	96.3	" "
Standard deviation	100	" "

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